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LANDESMAN-LAZER CONDITIONS FOR RESONANT p-LAPLACIAN PROBLEMS WITH JUMPING NONLINEARITIES

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Abstract. We study the existence of solutions of the p-Laplacian Dirichlet problem

$$-\phi_p(u')' = \lambda\phi_p(u) + h(x,u) + f(x,u,u'), \quad x \in (0,1), \tag{1}$$

$$u(0) = u(1) = 0, \tag{2}$$

where $\lambda \in \mathbb{R}$, $p > 1$, $\phi_s(\xi) := |\xi|^{s-1}\mathrm{sgn}\,\xi$ for $s \geqslant 1$, $\xi \in \mathbb{R}$, the function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ has the form

$$h(x,\xi) = a^{+\infty}(x)\phi_p(\xi^+) - a^{-\infty}(x)\phi_p(\xi^-), \quad (x,\xi) \in [0,1] \times \mathbb{R},$$

with $\xi^{\pm} := \max\{\pm\xi, 0\}$, and $a^{\pm\infty} \in L^1(0,1)$, and the function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and satisfies

$$|f(x,\xi,\eta)| \leqslant K(x)(1 + |\xi|^{q-1}), \quad (x,\xi,\eta) \in [0,1] \times \mathbb{R}^2,$$

for some $q \in [1,p)$ and $K \in L^1(0,1)$.

The dominant asymptotic behaviour of equation (1) as $u \to \pm\infty$ is determined by the coefficients $a^{\pm\infty}$, and we allow $a^- \neq a^+$, in which case the problem is said to be *jumping*. If the positively homogeneous problem obtained from (1)–(2) by setting $f \equiv 0$ has a non-trivial solution then the problem is said to be *resonant*, and λ is said to be a *half-eigenvalue*. Assuming that the problem (1)–(2) is both jumping and resonant, we will obtain a solution under certain 'Landesman-Lazer' conditions on f.

1. Introduction

We consider the p-Laplacian Dirichlet problem

$$-\phi_p(u')' = \lambda\phi_p(u) + h(x,u) + f(x,u,u'), \quad x \in (0,1), \tag{1}$$

$$u(0) = u(1) = 0, \tag{2}$$

where $\lambda \in \mathbb{R}$, $p > 1$, $\phi_s(\xi) := |\xi|^{s-1}\mathrm{sgn}\,\xi$ for $s \geqslant 1$, $\xi \in \mathbb{R}$, the function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ has the form

$$h(x,\xi) = a^{+\infty}(x)\phi_p(\xi^+) - a^{-\infty}(x)\phi_p(\xi^-), \quad (x,\xi) \in [0,1] \times \mathbb{R},$$

with $\xi^{\pm} := \max\{\pm\xi, 0\}$, and $a^{\pm\infty} \in L^1(0,1)$, and the function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and satisfies

$$|f(x,\xi,\eta)| \leqslant K(x)(1 + |\xi|^{q-1}), \quad (x,\xi,\eta) \in [0,1] \times \mathbb{R}^2, \tag{3}$$

for some $q \in [1,p)$ and $K \in L^1(0,1)$.

The dominant asymptotic behaviour of (1) as $u \to \pm\infty$ is determined by the coefficients $a^{\pm\infty}$ (and, of course, the values of p and λ), and we allow $a^- \neq a^+$, in which case the problem is said to be *jumping*. Clearly, if $t \geqslant 0$ then $h(x,t\xi) = t^{p-1}h(x,\xi)$, for $(x,\xi) \in [0,1] \times \mathbb{R}$, but if the problem is jumping then, in general, this will not be true for $t < 0$. In view of this, in the jumping case we will say that the function h is *positively homogeneous* (with respect to ξ). In addition, if we set $f \equiv 0$ in (1)-(2) then the resulting problem is *positively homogeneous*, in the sense that if u is a solution then tu is also a solution, for any $t \geqslant 0$, but in the jumping case this need not be true, in general, for $t < 0$. If the positively homogeneous problem obtained from (1)-(2) has no non-trivial solutions then the problem is said to be *nonresonant*. On the other hand, if the positively homogeneous problem has a non-trivial solution u, then λ is said to be a *half-eigenvalue* and (1)-(2) is said to be *resonant*.

It is well known that obtaining solutions for resonant problems is considerably more difficult than for nonresonant problems. Assuming that the problem (1)-(2) is both jumping and resonant, we will obtain a solution under certain 'Landesman-Lazer' conditions on f. We will state our main result in Theorem 3.1, and then compare this with previous results. Before this, we introduce some further basic notation, terminology and preliminary results.

2. Preliminary results

2.1. **Notation and definitions.** For $i = 0, 1$, $C^i[0,1]$ will denote the usual spaces of continuous and continuously differentiable functions, endowed with their usual sup-type norms $|\cdot|_i$, while $L^1(0,1)$ denotes the space of integrable functions, with norm $\|\cdot\|_1$, and $W^{1,1}(0,1)$ is the space of functions $w \in C^0[0,1]$ with distributional derivative $w' \in L^1(0,1)$.

If $g_1 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous then, for any $w \in C^1[0,1]$, we define $g(w) \in C^0[0,1]$ by

$$g_1(w)(x) := g(x, w(x), w'(x)), \quad x \in [0,1].$$

Clearly, this 'Nemitskii' mapping $w \to g_1(w) : C^1[0,1] \to C^0[0,1]$ is continuous. If $g_0 : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, we define a similar (continuous) Nemitskii mapping $w \to g_0(w) : C^0[0,1] \to C^0[0,1]$. We also define the Nemitskii mappings $w \to a^{\pm\infty}\phi_p(w)$, $h : C^0[0,1] \to L^1(0,1)$ in the obvious manner. When necessary, we will use the term 'Nemitskii mapping' to emphasize the distinction between these mappings and the underlying real-valued functions.

Next, we define $D_p \subset C^1[0,1]$ to be the set of functions $w \in C^1[0,1]$ such that $\phi_p(w') \in W^{1,1}(0,1)$, and we define $\Delta_p : D(\Delta_p) \to L^1(0,1)$ by

$$D(\Delta_p) := \{w \in D_p : w \text{ satisfies (2)}\},$$
$$\Delta_p(w) := \phi_p(w')', \quad w \in D(\Delta_p).$$

With the above notation we can now rewrite (1)-(2) as

$$-\Delta_p(u) = h(u) + f(u), \quad u \in D(\Delta_p). \tag{4}$$

2.2. **Positively homogeneous problems and half eigenvalues.** We first consider the positively homogeneous initial value problems

$$-\phi_p(v')' = \lambda\phi_p(v) + h(v), \quad \text{on } (0,1), \tag{5}$$
$$v(0) = 0, \quad v'(0) = \nu 1 \tag{6}$$

for each $\nu \in \{\pm\}$ (with the obvious interpretation of $\nu 1$). By [12, Theorem 5], we have the following result.

Lemma 2.1. *Any initial value problem on $[0,1]$ for the differential equation (5) has a unique solution $v \in D_p$ and, by this uniqueness, v has only simple zeros in $[0,1]$.*

By Lemma 2.1, the specific initial value problems (5)-(6) have unique solutions, which we denote by $\Psi_{\lambda,\nu}$, $\nu \in \{\pm\}$.

Next, we consider the positively homogeneous boundary value problem,

$$-\Delta_p(u) = \lambda\phi_p(u) + h(u), \quad u \in D(\Delta_p). \tag{7}$$

If (7) has a non-trivial solution u then λ is called a *half-eigenvalue* of (7), with corresponding *half-eigenfunction* u, and we define the *spectrum* of (7) to be the set

$$\Sigma_H := \{\lambda \in \mathbb{R} : (7) \text{ has a non-trivial solution}\}.$$

Of course, these quantities depend on the coefficients $a^{\pm\infty}$, but we will regard these as fixed throughout, so we will not indicate this dependence explicitly. By definition, λ is a half-eigenvalue of (7) if and only if $\Psi_{\lambda,\nu}(1) = 0$ for some $\nu \in \{\pm\}$, and $\Psi_{\lambda,\nu}$ is then a corresponding half-eigenfunction.

To describe the structure of the set Σ_H we introduce some further notation. For each integer $k \geqslant 0$, let $S_{k,\pm}$ denote the set of functions $u \in D(\Delta_p)$ having only simple zeros and exactly k such zeros in $(0,1)$, and such that $\pm u'(0) > 0$. By Lemma 2.1, any non-trivial solution u of (7) has only simple zeros, so $u \in S_{k,-} \cup S_{k,+}$ for some $k \geqslant 0$. The following description of the spectrum Σ_H was given in [13, Theorem 3.1].

Theorem 2.2. *For each $k \geqslant 0$, (7) has unique solutions $(\lambda, u) = (\lambda_{k,\pm}, u_{k,\pm}) \in \mathbb{R} \times S_{k,\pm}$ with $u'_{k,\pm}(0) = \pm 1$. All the half-eigenfunctions corresponding to $\lambda_{k,\pm}$ are of the form $tu_{k,\pm}$, with $t > 0$, and the spectrum $\Sigma_H = \bigcup_{k\geqslant 0}\{\lambda_{k,\pm}\}$. The half-eigenvalues are increasing, in the sense that*

$$k' > k \implies \lambda_{k',\nu'} > \lambda_{k,\nu}, \quad \text{for each } \nu',\, \nu \in \{\pm\}, \tag{8}$$

and $\lim_{k\to\infty} \lambda_{k,\pm} = \infty$. Furthermore, $\pm u_{0,\pm} > 0$ on $(0,1)$.

2.3. **A nondegeneracy condition.** Due to the degeneracy in the p-Laplacian differential equation (5) at the zeros of $\Psi_{\lambda,\nu}$ and $\Psi'_{\lambda,\nu}$, $\nu \in \{\pm\}$, we will require a further technical assumption on the behaviour of $a^{\pm\infty}$ near to these zeros (Remarks 6.2 and A.5 below describe the use of this nondegeneracy condition). We use the following notation: for $x \in [0,1]$ and $\delta > 0$, let $B_\delta(x) := \{y \in [0,1] : |y - x| < \delta\}$.

Condition $(ND)_\lambda$ For given $\lambda \in \mathbb{R}$, Condition $(ND)_\lambda$ holds if the following conditions hold.

(a) If $1 < p < 2$: there exists $r > 1/(p-1) > 1$ such that if $\Psi_{\lambda,\nu}(z) = 0$, then $a^{\pm\infty}|_{B_\delta(z)} \in L^r(B_\delta(z))$, for some $\delta = \delta_{\nu,z} > 0$.

(b) If $p > 2$: there exists $\gamma > 0$ such that if $\Psi'_{\lambda,\nu}(z) = 0$, then

$$\pm\Psi_{\lambda,\nu}(z) > 0 \implies \bar{\nu}(\lambda + a^{\pm\infty})|_{B_\delta(z)} \geqslant \gamma, \quad \text{a.e. on } B_\delta(z),$$

for some $\delta = \delta_{\nu,z} > 0$ and $\bar{\nu} = \bar{\nu}_{\nu,z} \in \{\pm\}$.

We note that $\Psi_{\lambda,\nu}$ and $\Psi'_{\lambda,\nu}$ have only finitely many zeros (see Lemma 2.1 and Remark A.2 (c)), so the assumptions in Condition $(ND)_\lambda$ only need to hold at a finite number of points. Also, in Condition $(ND)_\lambda$–(b), we observe that $\Psi'_{\lambda,\nu}(z) = 0 \implies \Psi_{\lambda,\nu}(z) \neq 0$.

3. The main results

For $x \in [0,1]$, let

$$\underline{F}^{\pm\infty}(x) := \liminf_{\xi\to\pm\infty} \frac{\inf_{\eta\in\mathbb{R}}\{f(x,\xi,\eta)\}}{\phi_q(\xi)}, \quad \overline{F}^{\pm\infty}(x) := \limsup_{\xi\to\pm\infty} \frac{\sup_{\eta\in\mathbb{R}}\{f(x,\xi,\eta)\}}{\phi_q(\xi)}.$$

It follows from (3) that

$$|\underline{F}^{\pm\infty}(x)| \leqslant K(x), \quad |\overline{F}^{\pm\infty}(x)| \leqslant K(x), \quad x \in [0,1], \tag{9}$$

so that $\underline{F}^{\pm\infty}$, $\overline{F}^{\pm\infty} \in L^1(0,1)$.

Theorem 3.1. *Suppose that $\lambda \in \{\lambda_{k,\min/\max}\}$ for some $k \geqslant 0$, and Condition $(ND)_\lambda$ holds. Suppose also that one of the following holds:*

(A) $\lambda_{k,\min} = \lambda_{k,\max}$ *and one of the following alternatives holds:*

$$\text{(A1)} \qquad \int_0^1 \left(|u_{k,\nu}^+|^q\, \overline{F}^{+\infty} + |u_{k,\nu}^-|^q\, \overline{F}^{-\infty}\right) < 0, \quad \textit{for each } \nu \in \{\pm\}; \tag{10}$$

$$\text{(A2)} \qquad \int_0^1 \left(|u_{k,\nu}^+|^q\, \underline{F}^{+\infty} + |u_{k,\nu}^-|^q\, \underline{F}^{-\infty}\right) > 0, \quad \textit{for each } \nu \in \{\pm\}. \tag{11}$$

(B) $\lambda_{k,\min} < \lambda_{k,\max}$ *and one of the following alternatives holds:*

$$\text{(B1)} \quad \textit{if } \lambda = \lambda_{k,\min}, \qquad \int_0^1 \left(|u_{k,\min}^+|^q\, \overline{F}^{+\infty} + |u_{k,\min}^-|^q\, \overline{F}^{-\infty}\right) < 0; \tag{12}$$

$$\text{(B2)} \quad \textit{if } \lambda = \lambda_{k,\max}, \qquad \int_0^1 \left(|u_{k,\max}^+)|^q\, \underline{F}^{+\infty} + |u_{k,\max}^-|^q\, \underline{F}^{-\infty}\right) > 0. \tag{13}$$

Then (4) *has a solution.*

Remark 3.2. The paper [10] discusses equation (4), and Theorem 3.1 above is similar to [10, Theorem 2.1]. However, the hypotheses here are considerably weaker than those in [10]. Specifically, [10] assumes that: $q = 1$; the function $f = f(x,\xi)$ is independent of η; the partial derivative f_ξ exists, and the functions f, f_ξ are bounded and continuous on $[0,1] \times \mathbb{R}$; the limits $\lim_{\xi\to\pm\infty} f(x,\xi)$ exist and are continuous. Hence, Theorem 3.1 above is considerably stronger than [10, Theorem 2.1].

3.1. **Comparison with previous results.** The solvability properties of (4) depend strongly on the position of λ with respect to the spectrum Σ_H. When $\lambda \notin \Sigma_H$ the problem (4) is nonresonant, and this case has been extensively studied, see [13] and the references therein for more information. Here, we are interested in the solvability of (4) when $\lambda \in \Sigma_H$, that is, in the resonant case. Existence conditions similar to the inequalities (10)–(13) in Theorem 3.1 have been used for problems at resonance in many different contexts. They were used in a paper by Landesman and Lazer [11], and have since been known as *Landesman-Lazer conditions*, but many papers since then have considered such problems. Reviews of these results, with extensive bibliographies, can be found in [5], [7] and [8]. The recent paper [10] also discussed much of the literature closely related to the problem considered here. For brevity, we will not repeat these surveys here. However, we briefly sketch the evolution of the assumptions imposed on the problem.

The early results concerned semilinear problems (that is, with $p = 2$) which were 'asymptotically linear', that is, with $a^{+\infty} = a^{-\infty}$ (and often with $q = 1$). Clearly, asymptotically linear problem are not jumping and the resonances occur at (linear) eigenvalues. Both nonresonant, and then resonant, problems were considered under these conditions, using various methods such as ODE techniques (as here), degree theory and variational methods, see [5], [7], [8], and the references therein, for further details. The original Landesman-Lazer paper dealt with this (resonant) case (in a PDE setting). Next, the Dancer-Fučík spectrum was introduced in [4] and [9], to deal with semilinear jumping problems. Again, nonresonant problems were tackled first (in particular, in [4]), and the resonant case was considered thereafter. The Dancer-Fučík spectrum requires the coefficients $a^{\pm\infty}$ to be constant – to deal with non-constant coefficients the concept of half-eigenvalues was introduced by Berestycki in [3]. Other, similar, ideas have also been introduced for this purpose, but these seem to be, essentially, equivalent to the half-eigenvalue formulation so are not described here. Again, the nonresonant case was considered first, and then the resonant case was tackled. At each stage of this progression, after the semilinear theory was developed the p-Laplacian theory was then developed along similar lines (that is, via eigenvalues, Fučík spectrum, half-eigenvalues) for nonresonant and then resonant problems.

Finally, it should be noted that Landesman-Lazer conditions are sufficient conditions, but are not necessary for the general case $p \neq 2$ (they are sharp when $p = 2$). Existence results have been obtained under other hypotheses, see [5], and also [6] for more recent results in this direction.

4. Proof of Theorem 3.1

To simplify the notation slightly we observe that by replacing $a^{\pm\infty}$ by $a^{\pm\infty} + \lambda$, we may suppose that $\lambda = 0$. With this supposition, from now on we will write $\Psi_\pm := \Psi_{0,\pm}$.

We will show that, for any $\tau \in \mathbb{R}$, there exists a solution $\psi(\tau) \in D_p$ of the initial value problem

$$-\phi_p(u')' = h(u) + f(u), \tag{14}$$

$$u(0) = 0, \quad u'(0) = |\tau|^{\frac{1}{p-q}} \operatorname{sgn} \tau \tag{15}$$

(in general, solutions of (14)-(15) need not be unique). Then, by definition, $\psi(\tau)$ satisfies (4) if and only if

$$\psi(\tau)(1) = 0, \tag{16}$$

so to prove the existence of a solution of (4) it suffices to show that there exists $\tau \in \mathbb{R}$ and a corresponding solution $\psi(\tau) \in D_p$ of (14)-(15) for which (16) holds. This will be done via the following two steps.

(*S*1) It will be shown that there exists a 'large' $\tau_0 > 0$ such that

$$\psi(\tau_0)(1)\, \psi(-\tau_0)(1) < 0. \tag{17}$$

To do this it will be convenient to first rescale the problem so that a 'large' τ corresponds to a 'small' $\widetilde{\tau}$, and then obtain the analogue of (17) for 'small' $\widetilde{\tau}$.

(*S*2) If the values $\psi(\tau)(1)$ were unique and depended continuously on $\tau \in [-\tau_0, \tau_0]$, then it would follow immediately from (17) that (16) holds for some $\tau \in (-\tau_0, \tau_0)$. Unfortunately, this deduction is not so simple since the solutions of the problems (14)-(15) need not be unique (so continuous dependence is not clear). However, we will use a connectedness property of the set of solutions of (14)-(15) to obtain (16) from (17).

These steps will be carried out in the following subsections.

4.1. Step (*S*1): a rescaled problem.

For any $\widetilde{\tau} > 0$, we consider the initial value problems

$$-\phi_p(u')' = h(u) + \widetilde{\tau}\widetilde{f}(u, \widetilde{\tau}), \tag{18}$$

$$u(0) = 0, \quad u'(0) = \pm 1, \tag{19}$$

where

$$\widetilde{f}(u, \widetilde{\tau}) := \widetilde{\tau}^{\frac{q-1}{p-q}} f(\widetilde{\tau}^{-\frac{1}{p-q}} u), \quad u \in D_p.$$

In the following two propositions we state some basic solution properties of (18)-(19). The proofs of these propositions will be postponed to Sections 5 and 6. These results are analogous to Propositions 2.1 and 2.2 in [10], but the hypotheses here are considerably weaker than those in [10], so we will give complete proofs of these results below.

Proposition 4.1. (*a*) *For any $\widetilde{\tau} > 0$, the problems* (18)-(19) *have solutions $\widetilde{\psi}_\pm(\widetilde{\tau}) \in D_p$ (these solutions need not be unique).*
(*b*) *There exists a decreasing sequence $(\widetilde{\tau}_n)$ in $(0, 1]$, with $\widetilde{\tau}_n \searrow 0$, such that*

$$\lim_{n\to\infty} \widetilde{\psi}_\pm(\widetilde{\tau}_n) = \Psi_\pm, \quad \textit{in } C^1[0,1]. \tag{20}$$

From now on (until we come to the proof of Proposition 4.1, in Section 5), (τ_n) will denote the sequence found in Proposition 4.1, or a subsequence thereof.

The solution mappings $\widetilde{\tau} \to \widetilde{\psi}_\pm(\widetilde{\tau})$ given by Proposition 4.1 need not be continuous, except for the weak form of continuity at $\widetilde{\tau} = 0$ in (20). The next proposition describes a similar weak differentiability at $\widetilde{\tau} = 0$. We define the difference quotients

$$Q_\pm(n) := \frac{1}{\widetilde{\tau}_n}\big\{\widetilde{\psi}_\pm(\widetilde{\tau}_n) - \Psi_\pm\big\} \in C^0[0,1], \quad n = 1, 2, \dots, \tag{21}$$

and we will show that these difference quotients converge as $n \to \infty$. The following notation will be used: for any $w \in C^0[0,1]$, let $\chi_{w^\pm}$ denote the characteristic functions of the sets $\{x \in [0,1] : w^\pm(x) > 0\}$. Also, the notation $\Sigma_\pm$ will denote summation over both the $+$ and $-$ forms of any succeeding terms containing the symbol $\pm$. So, for instance,

$$h(x,\xi) = \textstyle\sum_\pm a^{\pm\infty}(x)\phi_p(\xi^\pm), \quad (x,\xi) \in [0,1] \times \mathbb{R}.$$

Proposition 4.2. *For each $\nu \in \{\pm\}$, the following limits exist,*

$$Q^0_\nu := \lim_{n\to\infty} Q_\nu(n), \quad \textit{in } C^0[0,1]. \tag{22}$$

Also, there exists $M^{\pm\infty}_{\nu,f} \in L^1(0,1)$ satisfying

$$\underline{F}^{\pm\infty} \leqslant M^{\pm\infty}_{\nu,f} \leqslant \overline{F}^{\pm\infty}, \tag{23}$$

such that Q^0_ν satisfies the linear initial value problem

$$-\big(|\Psi'_\nu|^{p-2}\,(Q^0_\nu)'\big)' - \big(\textstyle\sum_\pm a^{\pm\infty}|\Psi^\pm_\nu|^{p-2}\chi_{\Psi^\pm_\nu}\big)\,Q^0_\nu = \frac{1}{p-1}\sum_\pm \pm M^{\pm\infty}_{\nu,f}\phi_q(\Psi^\pm_\nu), \tag{24}$$

$$Q^0_\nu(0) = 0, \quad (Q^0_\nu)'(0) = 0. \tag{25}$$

Furthermore, the coefficients in (24) *satisfy the standard assumptions on linear Sturm-Liouville problems, that is,*

$$|\Psi'_\nu|^{2-p} \in L^1(0,1) \quad \textit{and} \quad a^{\pm\infty}|\Psi^\pm_\nu|^{p-2}\chi_{\Psi^\pm_\nu} \in L^1(0,1), \tag{26}$$

so, in particular, the initial value problems (24), (25) *have unique solutions.*

Propositions 4.1 and 4.2 now enable us to complete step $(S1)$ described above. For any $\tau \neq 0$, let

$$\psi(\tau) := |\tau|^{\frac{1}{p-q}}\widetilde{\psi}_\pm(|\tau|^{-1}), \quad \text{if } \pm\tau > 0 \tag{27}$$

(where $\widetilde{\psi}_\pm$ are the solutions of (18)-(19) given by Proposition 4.1). It can be verified that $\psi(\tau)$ is a solution of (14)-(15). The following proposition will complete Step $(S1)$.

Proposition 4.3. *There exists $\tau_0 > 0$ such that the solutions $\psi(\pm\tau_0)$ given by* (27) *satisfy* (17).

Proof. We will show that

$$\widetilde{\psi}_+(\widetilde{\tau}_n)(1)\ \widetilde{\psi}_-(\widetilde{\tau}_n)(1) < 0, \quad \text{for sufficiently large } n, \tag{28}$$

and hence (17) holds with $\tau_0 := \widetilde{\tau}_n^{-1}$, for sufficiently large n. The proof is similar to the proof of [10, Lemma 3.5], albeit with some significant changes to the details of the formulae. Since this is the heart of the argument, we will write out the details.

Suppose that $\nu \in \{\pm\}$ is such that $\lambda_{k,\nu} = 0$. Then $u_{k,\nu} = \Psi_\nu$ and, by definition,

$$-\Delta_p(\Psi_\nu) - \textstyle\sum_\pm \pm a^{\pm\infty}\phi_p(\Psi^\pm_\nu) = 0, \tag{29}$$

$$\Psi_\nu(0) = 0, \quad \Psi'_\nu(0) = \nu 1, \quad \Psi_\nu(1) = 0, \quad \Psi'_\nu(1) \neq 0. \tag{30}$$

Multiplying (24) by Ψ_ν, integrating by parts, and using (23)-(30) yields

$$|\Psi_\nu'(1)|^{p-2}\Psi_\nu'(1)Q_\nu^0(1) = \frac{1}{p-1}\int_0^1 \textstyle\sum_\pm M_{\nu,f}^{\pm\infty}\,|\Psi_\nu^\pm|^q \neq 0 \tag{31}$$

(the final inequality follows from the hypotheses (10)-(13), together with (23), whichever case we are in). Hence, by (30) and (31), $\Psi_\nu(1) = 0$ and $Q_\nu^0(1) \neq 0$, so it follows from (21) and (22) that

$$\widetilde{\psi}_\nu(\widetilde{\tau}_n)(1)\,Q_\nu^0(1) > 0, \quad \text{for sufficiently large } n. \tag{32}$$

We now consider the two cases in the theorem.

Case (A). In this case $\lambda_{k,\pm} = 0$ and $\Psi_\pm(1) = 0$, so we must have $\Psi_-'(1)\,\Psi_+'(1) < 0$ (otherwise, by uniqueness of the solutions of initial value problems for (5), we would have $\Psi_+ = t\Psi_-$, for some $t > 0$, which would contradict $\Psi_+'(0) = -\Psi_-'(0)$). Combining this with (10)-(11), (23) and (31), yields $Q_-^0(1)Q_+^0(1) < 0$, and combining this with (32) (with both $\nu = \pm$) yields (28).

Case (B). Suppose that $\lambda_{k,+} = 0 > \lambda_{k,-}$ (the other cases are similar). Then [10, Lemma 2.1] shows that

$$\Psi_+'(1)\Psi_-(1) < 0, \tag{33}$$

so by (20),

$$\widetilde{\psi}_-(\widetilde{\tau}_n)(1)\,\Psi_-(1) > 0, \quad \text{for sufficiently large } n. \tag{34}$$

Combining (13), (23) and (31)-(34) (with $\nu = +$), now yields (28), which completes the proof of Proposition 4.3. □

4.2. **Step ($S2$): conclusion of the proof of Theorem 3.1.** To complete the proof of Theorem 3.1 we follow the strategy outlined in step ($S2$) above (using connectedness of the set of solutions values at $x = 1$). The argument for this is essentially identical to that in [10, Section 3.2], so will be omitted here. □

5. Proof of Proposition 4.1

(a) It follows from Theorem 1.3, Chap. 1, of [2] (see also Section 5, Chap. 1, of [2]) that for any $\widetilde{\tau} > 0$ the initial value problems (18)-(19) have a local solution $u_{\widetilde{\tau}}$ (possibly non-unique) on an interval $I_{u_{\widetilde{\tau}}} \subset [0,1]$ containing 0 (in this part of the proof we will suppose that one or other of the $\pm$ signs in (15) has been chosen and is fixed, so we omit these from the notation). We now define

$$m(x) := \max\{|u_{\widetilde{\tau}}(y)| : 0 \leqslant y \leqslant x\}, \quad x \in I_{u_{\widetilde{\tau}}}.$$

By integrating (18), and using (3) and (19), we see that

$$|\phi_p(u_{\widetilde{\tau}}'(x))| \leqslant 1 + C\big(1 + m(x)^{p-1} + \widetilde{\tau}m(x)^{q-1}\big), \quad x \in I_{u_{\widetilde{\tau}}}$$

(here, and below, C will denote a positive constant, which may be different on each occasion but which does not depend on $\widetilde{\tau}$ or $u_{\widetilde{\tau}}$), and hence

$$|u_{\widetilde{\tau}}'(x)| \leqslant CM_{\widetilde{\tau}}(1 + m(x)), \quad x \in I_{u_{\widetilde{\tau}}}, \tag{35}$$

where $M_{\widetilde{\tau}} := (1+\widetilde{\tau})^{\frac{1}{p-1}}$.

By (35) and a further integration,

$$m(x) \leqslant CM_{\widetilde{\tau}} + CM_{\widetilde{\tau}}\int_0^x m, \quad x \in I_{u_{\widetilde{\tau}}},$$

so by Gronwall's inequality and (35),

$$|u_{\widetilde{\tau}}(x)| + |u_{\widetilde{\tau}}'(x)| \leqslant CM_{\widetilde{\tau}}, \quad x \in I_{u_{\widetilde{\tau}}}. \tag{36}$$

It now follows from Theorem 4.1, Chap. 1, of [2] that the solution $u_{\widetilde{\tau}}$ may be extended to the whole of the interval $[0,1]$ (not necessarily in a unique manner). Hence, from now on we may suppose that any solution of (18)-(19) is defined on $[0,1]$ and belongs to D_p, so for each $\widetilde{\tau} > 0$ we may choose solutions $\widetilde{\psi}_\pm(\widetilde{\tau})$. This completes the proof of part (a).

(b) It will now be convenient to convert the initial value problems (18)-(19) into an integral operator formulation. By definition, the inverse of the Nemitskii mapping $\phi_p : C^0[0,1] \to C^0[0,1]$ can be written in the form $\phi_p^{-1} = \phi_{p^*+1}$, where $p^* := 1/(p-1)$. We now define operators $\mathcal{I} : L^1(0,1) \to W^{1,1}(0,1)$ and $\Gamma_\pm : L^1(0,1) \to C^1[0,1]$ by

$$\mathcal{I}(w)(x) := \int_0^x w(s)\,ds, \quad x \in (0,1), \quad \Gamma_\pm(w) := \mathcal{I}\big(\phi_{p^*+1}(\pm 1 + \mathcal{I}(w))\big), \quad w \in L^1(0,1).$$

These operators are continuous, and it can be verified that

$$\widetilde{\psi}_\pm(\widetilde{\tau}) = \Gamma_\pm\big[-h(\widetilde{\psi}_\pm(\widetilde{\tau})) - \widetilde{\tau}\widetilde{f}(\widetilde{\psi}_\pm(\widetilde{\tau}))\big], \quad \widetilde{\tau} > 0, \tag{37}$$

$$\Psi_\pm = \Gamma_\pm(-h(\Psi_\pm))). \tag{38}$$

From now on we suppose that $\widetilde{\tau} \in (0,1]$. Then, by (3), (36) and the definition of $\widetilde{f}$, there exists a constant $\widetilde{C} > 0$ such that

$$|\widetilde{f}(\widetilde{\psi}_\nu(\widetilde{\tau}_n), \widetilde{\tau}_n)(x)| \leqslant \widetilde{C}K(x), \quad x \in [0,1], \quad n = 1,2,\dots. \tag{39}$$

Now, by (36) and the compactness of the embedding of $C^1[0,1]$ into $C^0[0,1]$, we may choose a decreasing sequence $(\widetilde{\tau}_n)$ in $(0,1]$ such that $\widetilde{\tau}_n \to 0$ and $\widetilde{\psi}_\pm(\widetilde{\tau}_n) \to \widetilde{\psi}_{\pm,\infty}$ in $C^0[0,1]$, for some $\widetilde{\psi}_{\pm,\infty} \in C^0[0,1]$. It then follows from (37) and (39) that $\widetilde{\psi}_{\pm,\infty} = \Gamma_\pm(-h(\widetilde{\psi}_{\pm,\infty}))$, and hence $\widetilde{\psi}_{\pm,\infty}$ satisfy the initial value problems (5)-(6), so by the uniqueness of the solutions of these problems we must have $\widetilde{\psi}_{\pm,\infty} = \Psi_\pm$, and $\widetilde{\psi}_\pm(\widetilde{\tau}_n) \to \Psi_\pm$ in $C^1[0,1]$, which is (20). This completes the proof of Proposition 4.1. □

6. Proof of Proposition 4.2

Proposition 4.2 is similar to [10, Proposition 3.4], but there are also major differences between these results, due to the much weaker hypotheses imposed on f here (as detailed in Remark 3.2). In particular, it is assumed in [10] that the limits $\lim_{\xi\to\pm\infty} f(x,\xi)$ exist, whereas this is not assumed here. This makes it necessary to construct the functions $M_\nu^{+\infty}$ here, and makes it more difficult to obtain convergence of the difference quotients $Q_\nu(n)$. Due to these differences, the proof of Proposition 4.2 is significantly different to the proof of [10, Proposition 3.4]. In fact, we will use some of the machinery developed in [1] to deal with some of the additional problems caused by the weaker hypotheses on f.

We first construct the required functions $M_{\nu,f}^{\pm\infty}$.

Lemma 6.1. *There exists a decreasing sequence (τ_n) such that $\tau_n \to 0$, and for each $\nu \in \{\pm\}$ there exist functions $M_{\nu,f}^{\pm\infty} \in L^1(0,1)$ satisfying (23), such that*

$$\mathcal{I}\big(\widetilde{f}(\widetilde{\psi}_\nu(\widetilde{\tau}_n), \widetilde{\tau}_n)\big) \to \mathcal{I}\big(\textstyle\sum_\pm \pm M_{\nu,f}^{\pm\infty}\phi_q(\Psi_\nu^\pm)\big), \quad \text{in } C^0[0,1]. \tag{40}$$

Proof. It follows from (39) and [13, Lemma 2.1] that there exists a subsequence of the sequence constructed in Proposition 4.1 (which we continue to denote by (τ_n)) and functions $\Xi_\pm \in L^1(0,1)$ such that

$$\widetilde{f}(\widetilde{\psi}_\pm(\widetilde{\tau}_n), \widetilde{\tau}_n) \rightharpoonup \Xi_\pm, \qquad \text{in } L^1(0,1),$$

$$\mathcal{I}\big(\widetilde{f}(\widetilde{\psi}_\pm(\widetilde{\tau}_n), \widetilde{\tau}_n)\big) \to \mathcal{I}(\Xi_\pm), \qquad \text{in } C^0[0,1],$$

where $\rightharpoonup$ denotes weak convergence in $L^1(0,1)$. Now, for either $\nu \in \{\pm\}$, suppose that $A \subset [0,1]$ is an arbitrary closed interval on which $\Psi_\nu > 0$ (recall that, by Lemma 2.1, Ψ_ν has only simple zeros in $[0,1]$). Then, by (20), there exists $\delta_A > 0$ such that $\widetilde{\psi}_\nu(\widetilde{\tau}_n) \geqslant \delta_A$ on A, for all sufficiently large n, so by weak convergence in $L^1(0,1)$, the definitions of $\widetilde{f}$ and $\overline{F}^{+\infty}$, and Fatou's lemma,

$$\int_A \Xi_\nu = \lim_{n\to\infty} \int_A \widetilde{f}(\widetilde{\psi}_\nu(\widetilde{\tau}_n), \widetilde{\tau}_n) \leqslant \int_A \overline{F}^{+\infty} \phi_p(\Psi_\nu).$$

Since A was arbitrary, we conclude that $\Xi_\nu \leqslant \overline{F}^{+\infty} \phi_p(\Psi_\nu)$ a.e. on the set where $\Psi_\nu > 0$. A similar argument shows that $\Xi_\nu \geqslant \underline{F}^{+\infty} \phi_p(\Psi_\nu)$ on this set, and similar bounds can be obtained on the set where $\Psi_\nu < 0$. Combining these bounds then shows that the functions

$$M^{\pm\infty}_{\nu,f} := \pm \frac{\Xi_\nu}{\phi_p(\Psi^\pm_\nu)} \, \chi_{\Psi^\pm_\nu} \in L^1(0,1),$$

are well-defined, and satisfy (23) on the set $\{x \in [0,1] : \Psi_\nu \neq 0\}$, and since Ψ_ν has only simple zeros, we may suppose that (23) holds on $[0,1]$, which completes the proof of Lemma 6.1. □

We now prove (22). For each $\nu \in \{\pm\}$ the function Ψ_ν has only simple zeros, so combining (37), (38), and Lemmas 6.1, A.3 and A.4 (see Remark 6.2), shows that, for $n = 1, 2, \dots$,

$$\begin{aligned}
Q_\nu(n) &= \frac{1}{\tau_n} \big\{ \Gamma_\nu\big[- h(\widetilde{\psi}_\nu(\widetilde{\tau}_n) - \widetilde{\tau}_n \widetilde{f}(\widetilde{\psi}_\nu(\widetilde{\tau}_n), \widetilde{\tau}_n)\big] - \Gamma_\nu[-h(\Psi_\nu)] \big\} \\
&= \frac{1}{\tau_n} D\Gamma_\nu(w_\nu) \, \big\{ - \textstyle\sum_\pm \pm a^{\pm\infty} \big(\phi_p(\widetilde{\psi}_\nu(\widetilde{\tau}_n))^\pm - \phi_p(\Psi_\nu)^\pm \big) - \widetilde{\tau}_n \widetilde{f}(\widetilde{\psi}_\nu(\widetilde{\tau}_n), \widetilde{\tau}_n) \big\} + R_1(n) \\
&= D\Gamma_\nu(w_\nu) \, \big\{ - (p-1) \textstyle\sum_\pm a^{\pm\infty} |\Psi^\pm_\nu|^{p-2} \chi_{\Psi^\pm_\nu} \, Q_\nu(n) - \widetilde{f}(\widetilde{\psi}_\nu(\widetilde{\tau}_n), \widetilde{\tau}_n) \big\} + R_2(n),
\end{aligned}$$

where $R_1(n)$, $R_2(n) \in C^0[0,1]$, satisfy

$$\begin{aligned}
&|R_1(n)|_0 = \mathrm{o}(1) \, \tau_n^{-1} \textstyle\sum_\pm \big| \big(\phi_p(\widetilde{\psi}_\nu(\widetilde{\tau}_n))^\pm - \phi_p(\Psi_\nu)^\pm \big|_0 + \mathrm{o}(1) \, \big| \widetilde{f}(\widetilde{\psi}_\nu(\widetilde{\tau}_n), \widetilde{\tau}_n) \big) \big|_0, \\
&|R_2(n)|_0 = \mathrm{o}(1) \, \big(|Q_\nu(n)|_0 + 1 \big),
\end{aligned}$$

as $n \to \infty$. It follows from this and Gronwall's inequality (using the form of the operators $D\Gamma_\nu(w_\nu)$ given in (48)) that $|Q_\nu(n)|_0$ is bounded as $n \to \infty$, and since $D\Gamma_\nu(w_\nu) : L^1(0,1) \to C^0[0,1]$ is compact we may suppose that the limits $Q^0_\nu = \lim_{n\to\infty} Q_\nu(n)$ exist, in $C^0[0,1]$. Hence, letting $n \to \infty$ in the above calculation, and using (40), shows that

$$Q^0_\nu = D\Gamma_\nu(w_\nu) \, \big\{ - (p-1) \textstyle\sum_\pm a^{\pm\infty} |\Psi^\pm_\nu|^{p-2} \chi_{\Psi^\pm_\nu} \, Q^0_\nu - \textstyle\sum_\pm \pm M^{\pm\infty}_{\nu,f} \, \phi_q(\Psi^\pm_\nu) \big\}. \tag{41}$$

Also, combining (41) with (46) shows that Q^0_ν satisfies (24)-(25).

Finally, we prove (26).

- $a^{\pm\infty} |\Psi^\pm_\nu|^{p-2} \chi_{\Psi^\pm_\pm} \in L^1(0,1)$: this is obvious if $p \geqslant 2$, while if $1 < p < 2$ then it follows from Condition $(ND)_0$–(a) and Lemma A.3-(b), since Ψ_ν has only simple zeros;
- $|\Psi'_\nu|^{2-p} \in L^1(0,1)$: this is obvious if $1 < p \leqslant 2$, while if $p > 2$ then, by Remarks A.2 (b) and (c),

$$|\Psi'_\nu|^{2-p} = |\phi_p(\Psi'_\nu)|^{-\frac{p-2}{p-1}} \in L^1(0,1).$$

This completes the proof of Proposition 4.2. □

Remark 6.2. The above proof relied on Lemmas A.3 and A.4 to differentiate the Nemitskii mappings $a^{\pm\infty}\phi^\pm_p$ at Ψ_ν, and the operators Γ_ν at $-h(\Psi_\nu)$, respectively. Lemma 2.1 and Condition $(ND)_0$–(a) ensure that the hypotheses of Lemma A.3 hold at Ψ_ν. Lemma A.4 is stated as holding at $-h(\Psi_\nu)$, and its proof relied on Condition $(ND)_0$–(b), see Remark A.5.

Appendix A. Some differentiability results

Definition A.1. If $w \in W^{1,1}(0,1)$ and $w(z) = 0$ then we say that z is a *simple zero* of w if there exists positive numbers δ_z, γ_z, and $\bar\nu_z \in \{\pm\}$, such that

$$\bar\nu_z \frac{w(x)}{x-z} \geqslant \gamma_z, \quad x \in B_{\delta_z}(z) \setminus \{z\}. \tag{42}$$

Remark A.2. (a) A natural sufficient condition for (42) to hold is that $\bar\nu_z w'(x) \geqslant \gamma_z$, for a.e. $x \in B_{\delta_z}(z)$, that is, if the derivative $w' \in L^1(0,1)$ is uniformly bounded away from zero, and has only one sign near to z. This is clearly akin to the standard idea of a simple zero of $w \in C^1[0,1]$.
(b) If $w \in W^{1,1}(0,1)$ has only simple zeros in $[0,1]$ then (42) implies that $w^{-\alpha} \in L^1(0,1)$, for any $0 < \alpha < 1$.
(c) If $p > 2$ and $\Psi'_\nu(z) = 0$, for some $\nu \in \{\pm\}$ and $z \in [0,1]$, then by Lemma 2.1, $\phi_p(\Psi_\nu(z)) \neq 0$, so by Condition $(ND)_0$–(b), the differential equation (29) (which holds for each $\nu \in \{\pm\}$), and remark (a), the function $\phi_p(\Psi'_\nu) \in W^{1,1}(0,1)$ has only simple zeros in $[0,1]$.

Lemma A.3. *Suppose that $a \in L^1(0,1)$, $w_0 \in C^0[0,1]$, and either (a) or (b) below holds. Then $a|w_0|^{p-2} \in L^1(0,1)$ and the Nemitskii mappings $a\phi_p$, $a\phi_p^\pm : C^0[0,1] \to L^1(0,1)$ are differentiable at $w_0 \in C^0[0,1]$, with derivatives given by*

$$D(a\phi_p)(w_0)\,\bar w = (p-1)a|w_0|^{p-2}\,\bar w, \quad D(\pm a\phi_p^\pm)(w_0)\,\bar w = (p-1)a|w_0|^{p-2}\chi_{w_0^\pm}\,\bar w, \quad \bar w \in C^0[0,1], \tag{43}$$

- (a) *$p \geqslant 2$ and $w_0 \in C^0[0,1]$ is arbitrary.*
- (b) *$1 < p < 2$ and $w_0 \in W^{1,1}(0,1)$ has only simple zeros in $[0,1]$, and there exists $r_0 > 1/(p-1) > 1$ such that, for every zero z of w_0, there exists $\delta_z > 0$ such that $a|_{B_{\delta_z}(z)} \in L^{r_0}(B_{\delta_z}(z))$.*

Proof. (a) When $a \equiv 1$ this follows immediately from the method of proof of [1, Theorem 3.2]. The extension to deal with general $a \in L^1(0,1)$ is straightforward in this case.

(b) Define s_0 by $1/r_0 + 1/s_0 = 1$. Clearly $1 < s_0 < 1/(2-p)$, and if z is a (simple) zero of w_0 and δ_z is sufficiently small then $|w_0|^{p-2} \in L^{s_0}(B_{\delta_z}(z))$. Combining this with the hypothesis on a shows that $a|w_0|^{p-2} \in L^1(B_{\delta_z}(z))$, and hence, since w_0 has only simple zeros, $a|w_0|^{p-2} \in L^1(0,1)$. It follows immediately from this that $a|w_0|^{-1+(p-1)/2} \in L^1(0,1)$, and combining this with [1, Lemma 3.3] shows that

$$\big\|a\phi_p(w_0+\bar w) - a\phi_p(w_0) - (p-1)a|w_0|^{p-2}\,\bar w\big\|_1 \leqslant C|\bar w|_0^{1+(p-1)/2}\big\|a|w_0|^{-1+(p-1)/2}\big\|_1,$$

so letting $|\bar w|_0 \to 0$ proves the differentiability result for the Nemitskii mapping $a\phi_p$.

Next, since w_0 has only simple zeros it can be seen that for sufficiently small $\bar w \in C^0[0,1]$,

$$\pm\big(\phi_p(w_0+\bar w)^\pm - \phi_p(w_0)^\pm\big) = \big(\phi_p(w_0+\bar w) - \phi_p(w_0)\big)\chi_{w_0^\pm} + R_\pm(\bar w),$$

where $R_\pm(\bar w) \in C^0[0,1]$, with $|R_\pm(\bar w)|_0 \leqslant C_1|\bar w|_0^{p-1}$, and the set

$$Z_{R_\pm} := \{x \in [0,1] : R_\pm(\bar w)(x) \neq 0\} \subset \bigcup_{z \in w_0^{-1}(0)} B_{C_2|\bar w|_0}(z)$$

(that is, $Z_{R_\pm}$ is contained in a union of intervals surrounding the zeros of w_0, with length $C_2|\bar w|_0$). Hence,

$$\begin{aligned} \|\pm\big(a\phi_p(w_0+\bar w)^\pm - &a\phi_p(w_0)^\pm\big) - (p-1)a|w_0|^{p-2}\chi_{w_0^\pm}\bar w\|_1 \\ &\leqslant \|a\phi_p(w_0+\bar w) - a\phi_p(w_0) - (p-1)a|w_0|^{p-2}\bar w\|_1 + \|aR_\pm(\bar w)\|_1. \end{aligned} \tag{44}$$

Now, applying Hölder's inequality on the interval $B_{C_2|\bar{w}|_0}(z)$, for each zero z of w_0, and using the hypothesis on a, shows that

$$\|aR_\pm(\bar{w})\|_1 \leqslant C_1|\bar{w}|_0^{p-1}\left(\int_0^{C_2|\bar{w}|_0} 1\right)^{1/s_0} = C|\bar{w}|_0^{p-1+1/s_0}. \tag{45}$$

Since $p-1+1/s_0 > 1$, combining (44) and (45) with the differentiability result for the Nemitskii mapping $a\phi_p$, now proves the differentiability result for the Nemitskii mappings $a\phi_p^\pm$. □

Lemma A.4. *For any $p>1$ and $\nu \in \{\pm\}$, the operator $\Gamma_\nu : L^1(0,1) \to C^0[0,1]$, is differentiable at*

$$w_\nu := -h(\Psi_\nu) = -\textstyle\sum_\pm \pm a^{\pm\infty}\phi_p(\Psi_\nu^\pm).$$

The derivative $D\Gamma_\nu(w_\nu)$ has the property that, for $\bar{w} \in L^1(0,1)$,

$$\overline{W} = D\Gamma_\nu(w_\nu)\,\bar{w} \iff \begin{cases} (p-1)\big(|\Psi_\nu'|^{p-2}\overline{W}'\big)' = \bar{w}, \\ \overline{W}(0) = \overline{W}'(0) = 0 \end{cases} \tag{46}$$

(*recall that $|\Psi_\nu'|^{2-p} \in L^1(0,1)$, by* (26)).

We note that Γ_ν maps $L^1(0,1)$ into $C^1[0,1]$, but Lemma A.4 does not seem to hold when we regard Γ_ν as a mapping into $C^1[0,1]$, so we have used $C^0[0,1]$ as the codomain in the lemma (the proof would also work with $W^{1,1}(0,1)$).

Proof. Differentiating (38) (with respect to x) and applying ϕ_p yields

$$\phi_p(\Psi_\pm') = \pm 1 + \mathcal{I}(w_\pm). \tag{47}$$

Now, by the form of the operators $\Gamma_\pm$, the linearity of $\mathcal{I}$, Lemma A.3, and (47), we see that $\Gamma_\pm$ are differentiable at $w_\pm$, and that

$$\begin{aligned} \overline{W} = D\Gamma_\pm(w_\nu)\,\bar{w} &= \mathcal{I}\circ D\phi_{p^*+1}\big(\pm 1 + \mathcal{I}(w_\pm)\big)\circ \mathcal{I}(\bar{w}) = \mathcal{I}\big(p^*|\phi_p(\Psi_\pm')|^{p^*-1}\,\mathcal{I}(\bar{w})\big) \\ &= \mathcal{I}\big(p^*|\Psi_\pm'|^{2-p}\,\mathcal{I}(\bar{w})\big), \end{aligned} \tag{48}$$

recalling that $p^* = 1/(p-1)$, and hence $p^*-1 = (2-p)/(p-1)$. It can now be verified that (48) is equivalent to the right hand side of (46). □

Remark A.5. In (48), Lemma A.3 was used to differentiate the Nemitskii mapping ϕ_{p^*+1}, at $\phi_p(\Psi_\pm')$. When $p^*+1 \geqslant 2$, i.e., when $p \leqslant 2$, part (a) of Lemma A.3 applies immediately. However, when $p^*+1 < 2$, i.e., when $p>2$, in order to apply part (b) of Lemma A.3 (with $a \equiv 1$) we need $\phi_p(\Psi_\pm')$ to have simple zeros. This follows from Condition $(ND)_0$–(b), see Remark A.2-(c).

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